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Solution of a Diffraction Problem II. The Narrow Double Wedge

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SOLUTION OF A DIFFRACTION PROBLEM
II. THE NARROW DOUBLE WEDGE

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The problem of diffraction by a 'narrow double wedge' (width much smaller than wavelength) is investigated. Strong reflexion and quasi-static effects are the main features of this problem. The asymptotic behaviour of the solution is determined by the edge singularities. This leads to an approximate solution, which seems to be very accurate. This solution is found to be in good agreement with approximate solutions derived by different methods. The reflexion coefficient and the 'end correction' are evaluated. The results are compared with those obtained by other authors. It is shown that they contain a new effect, the 'evanescent mode correction', which is very small in this region. Resonance effects in channels of finite length are analyzed.

1. Introduction

In part I of this paper (Nussenzveig 1959*b*; henceforward quoted as I), we have studied the problem of diffraction by a wide double wedge (width of double wedge much larger than wavelength). In this part, we shall consider the case of a *narrow double wedge* (width of double wedge much smaller than wavelength), i.e. we shall assume that $K \ll 1$; the notation is the same as in I.

For $K \ll 1$, the only travelling mode within the waveguide is the principal mode, which will therefore be taken as incident mode; all other modes are evanescent. According to I, (2·8), it follows that

$$u_{\mathrm{I}}(x, y) = \exp(-\mathrm{i}kx) + a_0 \exp(\mathrm{i}kx) + \sum_{n=1}^{\infty} a_n \cos(k_{y,n} y) \exp[-(k_{y,n}^2 - k^2)^{\frac{1}{2}} x]. \qquad (1·1)$$

The reflexion coefficient is

$$|R|^2 = |a_0|^2. \qquad (1·2)$$

As was pointed out in I, § 5 (g), the narrow double-wedge problem is a particular case of critical incidence; this is connected with the fact that propagation is no longer possible for $K \to 0$. This suggests that *strong reflexion* should occur in this domain.

Quasi-static effects should also become important for $K \ll 1$. In particular, the influence of the edge singularities extends over the whole aperture. According to I, § 4 (d), this determines the behaviour of the mode amplitudes a_n for $\gamma_n \gg 1$, i.e. the behaviour of all evanescent modes.

The above considerations allow us to predict at once the character of the solution: a_0 should be close to -1, and the main term of a_n, for $n \geqslant 1$, should be proportional to $n^{-\frac{5}{3}}$ (see I, § 4(d)). This will be confirmed by the detailed solution.

2. The coefficients $K_{m,n}$

For $K \ll 1$, the functions $S_n(K)$ and $C_n(K)$ defined in I, (3·6) and I, (3·9) may be expanded in series of powers and logarithms of K. For this purpose, it suffices to replace $H_0(v)$ by its expansion about the origin, and to integrate term by term (Nussenzveig 1957; henceforward quoted as A). The results are:

$$S_0(K) = 0, \tag{2·1}$$

$$C_0(K) = \frac{4i}{\pi} K \ln K + 2\left[1 - \frac{2i}{\pi}(1-C)\right]K - \frac{4i}{3\pi}K^3 \ln K - \frac{2}{3}\left[1 - \frac{2i}{\pi}(\tfrac{4}{3} - C)\right]K^3 + \frac{i}{5\pi}K^5 \ln K + O(K^5), \tag{2·2}$$

$$S_n(K) = -\frac{2i}{\pi}\frac{\tau_n}{n\pi}K + \frac{2i}{\pi}\frac{1}{n\pi}K^3 \ln K + \left\{\left[1 - \frac{2i}{\pi}(1-C)\right]\frac{1}{n\pi} - \frac{i\tau_n}{\pi(n\pi)^3}\right\}K^3 - \frac{i}{2\pi}\left[\frac{1}{n\pi} - \frac{3}{(n\pi)^3}\right]K^5 \ln K + O(K^5) \quad (n \geqslant 1), \tag{2·3}$$

$$C_n(K) = -\frac{2i}{\pi}\frac{\mathrm{Si}(2\pi n)}{n\pi}K - \frac{2i}{\pi}\frac{1}{(n\pi)^2}K^3 \ln K - \left\{\left[1 + \frac{i}{\pi}(2C-1)\right]\frac{1}{(n\pi)^2} + \frac{i}{\pi}\frac{\mathrm{Si}(2\pi n)}{(n\pi)^3}\right\}K^3 + \frac{i}{\pi}\left[\frac{1}{(n\pi)^2} - \frac{3}{2}\frac{1}{(n\pi)^4}\right]K^5 \ln K + O(K^5) \quad (n \geqslant 1), \tag{2·4}$$

where $C = 0·5772$ (Euler's constant), $\mathrm{Si}(x)$ is the sine integral, $\mathrm{Ci}(x)$ is the cosine integral, and

$$\tau_n = \ln(2\pi n) + C - \mathrm{Ci}(2\pi n). \tag{2·5}$$

Substituting these expressions in I, (3·12) to (3·14), we find

$$(-1)^{m+n}K_{m,n} = \frac{1}{\pi^2}\left[\frac{n(\tau_m - \tau_n)}{m^2 - n^2} - \frac{1}{2\pi^2}\frac{\tau_m}{m^2 n}K^2 + \frac{3}{4\pi^2}\frac{1}{m^2 n}K^4 \ln K + O(K^4)\right] \quad (m \neq n \neq 0), \tag{2·6}$$

$$K_{n,n} = \frac{1}{2} - \frac{1}{\pi}\mathrm{Si}(2\pi n) - \frac{1}{2\pi}\frac{\tau_n}{(n\pi)^3}K^2 + \frac{3}{4\pi}\frac{1}{(n\pi)^3}K^4 \ln K + O(K^4) \quad (n \neq 0), \tag{2·7}$$

$$(-1)^m K_{m,0} = -\frac{i}{\pi}\frac{\tau_m}{(m\pi)^2}K + \frac{i}{\pi}\frac{1}{(m\pi)^2}K^3 \ln K + \left\{\left[\frac{1}{2} - \frac{i}{\pi}(1-C)\right]\frac{1}{(m\pi)^2} - \frac{i}{2\pi}\frac{\tau_m}{(m\pi)^4}\right\}K^3 - \frac{i}{4\pi}\left[\frac{1}{(m\pi)^2} - \frac{3}{(m\pi)^4}\right]K^5 \ln K + O(K^5) \quad (m \neq 0), \tag{2·8}$$

$$(-1)^n K_{0,n} = \frac{1}{2\pi}\left\{\frac{\tau_n}{n\pi} - \frac{1}{n\pi}K^2 \ln K + \left(1 - C + \mathrm{i}\frac{\pi}{2}\right)\frac{K^2}{n\pi} + \frac{1}{4}\left[\frac{1}{n\pi} - \frac{1}{(n\pi)^3}\right]K^4 \ln K + O(K^4)\right\} \quad (n \neq 0), \tag{2·9}$$

$$K_{0,0} = \frac{1}{2} - \frac{\mathrm{i}}{\pi}K \ln K - \left[\frac{1}{2} - \frac{\mathrm{i}}{\pi}\left(\tfrac{3}{2} - C\right)\right]K + \frac{\mathrm{i}}{6\pi}K^3 \ln K + \frac{1}{12}\left[1 + \frac{\mathrm{i}}{\pi}\left(2C - \tfrac{19}{6}\right)\right]K^3 + O(K^5 \ln K). \tag{2·10}$$

We shall write these expressions in the form

$$(-1)^{m+n} K_{m,n} = K^{(0)}_{m,n} + K^{(1)}_{m,n}K^2 + K^{(2)}_{m,n}K^4 \ln K + \ldots \quad (m \neq 0, n \neq 0), \tag{2·11}$$

$$(-1)^m K_{m,0} = K^{(0)}_{m,0}K + K^{(1)}_{m,0}K^3 \ln K + K^{(2)}_{m,0}K^3 + \ldots \quad (m \neq 0), \tag{2·12}$$

$$(-1)^n K_{0,n} = K^{(0)}_{0,n} + K^{(1)}_{0,n}K^2 \ln K + K^{(2)}_{0,n}K^2 + \ldots \quad (n \neq 0), \tag{2·13}$$

$$K_{0,0} = \tfrac{1}{2} + K^{(0)}_{0,0}K \ln K + K^{(1)}_{0,0}K + K^{(2)}_{0,0}K^3 \ln K + \ldots, \tag{2·14}$$

where $K^{(0)}_{m,n}, K^{(1)}_{m,n}, \ldots$ may be obtained at once by comparison with (2·6) to (2·10).

3. The asymptotic method of solution

It follows from (2·8) and (2·10) that $\lim_{K\to 0}(K_{m,0}) = \tfrac{1}{2}\delta_{m,0}$. According to I, § 5 (*g*), this implies that $\lim_{K\to 0} a_n = -\delta_{n,0}$. Similarly, if we neglect terms of higher order than K in (2·11) to (2·14), it may be shown, as in I, § 5 (*g*), that system (S) separates into a system (S'), containing only the amplitudes of the evanescent modes, and a single equation which contains a_0.

To obtain higher-order approximations, we shall take an 'Ansatz' for the solution in the form of a series of powers and logarithms of K:

$$a_0 = -1 + a^{(0)}_0 K \ln K + a^{(1)}_0 K + a^{(2)}_0 K^2 (\ln K)^2 + a^{(3)}_0 K^2 \ln K + a^{(4)}_0 K^2 + a^{(5)}_0 K^3 (\ln K)^3 + a^{(6)}_0 K^3 (\ln K)^2 + a^{(7)}_0 K^3 \ln K + O(K^3), \tag{3·1}$$

$$(-1)^n a_n = a^{(0)}_n K + a^{(1)}_n K^2 \ln K + a^{(2)}_n K^2 + a^{(3)}_n K^3 (\ln K)^2 + a^{(4)}_n K^3 \ln K + O(K^3) \quad (n \geqslant 1). \tag{3·2}$$

Substituting (2·11), (2·12), (3·1) and (3·2) in system (S), for $m \geqslant 1$, and identifying, we obtain a series of infinite systems of linear equations (S_r) in the unknowns $a^{(r)}_n$ $(r = 0, 1, 2, \ldots)$. All these systems are of the form

$$(S_r) \quad a^{(r)}_m = \sum_{n=1}^{\infty} K^{(0)}_{m,n} a^{(r)}_n + b^{(r)}_m \quad (m = 1, 2, 3, \ldots), \tag{3·3}$$

and they differ from one another only by the inhomogeneous term $b^{(r)}_m$:

$$\left.\begin{aligned} b^{(0)}_m &= -2K^{(0)}_{m,0}, \quad b^{(1)}_m = -\tfrac{1}{2}a^{(0)}_0 b^{(0)}_m, \quad b^{(2)}_m = -\tfrac{1}{2}a^{(1)}_0 b^{(0)}_m, \\ b^{(3)}_m &= -\tfrac{1}{2}a^{(2)}_0 b^{(0)}_m, \quad b^{(4)}_m = a^{(3)}_0 K^{(0)}_{m,0} - 2K^{(1)}_{m,0}. \end{aligned}\right\} \tag{3·4}$$

It follows from (3·4) that

$$a^{(1)}_m = -\tfrac{1}{2}a^{(0)}_0 a^{(0)}_m; \quad a^{(2)}_m = -\tfrac{1}{2}a^{(1)}_0 a^{(0)}_m; \quad a^{(3)}_m = -\tfrac{1}{2}a^{(2)}_0 a^{(0)}_m. \tag{3·5}$$

Similarly, if we substitute (2·13), (2·14), (3·1) and (3·2) in the first equation of system (S) (for $m = 0$), and identify, we obtain a series of equations (E_s) in the coefficients $a_0^{(s)}$ $(s = 0, 1, \ldots)$.

In each order of approximation, we have to solve a system (S_r) and an equation (E_s). It is readily verified that the solution of each system (S_r) requires only the solution of equations (E_s) belonging to previous approximations, and vice versa. This means that *system* (S) *separates in all orders of approximation.*

Replacing the coefficients $K_{m,n}^{(r)}$ by their values, and taking into account systems (S_r), for $0 \leqslant r \leqslant 4$, and equations (E_s), for $0 \leqslant s \leqslant 7$, we finally get

$$a_0 = -1 + \frac{4i}{\pi} K \ln K + a_0^{(1)} K + \frac{8}{\pi^2} K^2 (\ln K)^2 - \frac{4i}{\pi} a_0^{(1)} K^2 \ln K - \tfrac{1}{2} (a_0^{(1)})^2 K^2$$
$$- \frac{16i}{\pi^3} K^3 (\ln K)^3 - \frac{12}{\pi^2} a_0^{(1)} K^3 (\ln K)^2 + a_0^{(7)} K^3 \ln K + O(K^3), \quad (3·6)$$

$$(-1)^n a_n = \left[1 - \frac{2i}{\pi} K \ln K - \tfrac{1}{2} a_0^{(1)} K - \frac{4}{\pi^2} K^2 (\ln K)^2\right] K a_n^{(0)}$$
$$+ a_n^{(4)} K^3 \ln K + O(K^3) \quad (n \geqslant 1), \quad (3·7)$$

where
$$a_0^{(1)} = 2\left[1 - \frac{2i}{\pi} (\tfrac{3}{2} - C)\right] + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \tau_n \frac{a_n^{(0)}}{n}, \quad (3·8)$$

$$a_0^{(7)} = -\frac{2i}{3\pi} + \frac{i}{\pi} (a_0^{(1)})^2 + \frac{4i}{\pi}\left[1 - \frac{2i}{\pi} (\tfrac{3}{2} - C)\right] a_0^{(1)} - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{a_n^{(0)}}{n} + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \tau_n \frac{a_n^{(4)}}{n}. \quad (3·9)$$

Equations (3·6) to (3·9) reduce the solution of the narrow double-wedge problem, up to terms of the order of K^3, to the solution of systems (S_0) and (S_4). According to (2·6), (2·7), (2·11) and (3·3), the coefficients of these systems are

$$K_{m,n}^{(0)} = \frac{n}{\pi^2} \frac{(\tau_m - \tau_n)}{(m^2 - n^2)} \quad (m \neq n); \qquad K_{n,n}^{(0)} = \frac{1}{2} - \frac{1}{\pi} \operatorname{Si}(2\pi n). \quad (3·10)$$

For $n \geqslant 1$, Si $(2\pi n)$ and Ci $(2\pi n)$ may be replaced by their asymptotic expansions (Bromwich 1949, p. 339), so that (2·5) becomes

$$\tau_n = \ln(2\pi n) + C + \frac{1}{(2\pi n)^2} - \frac{3!}{(2\pi n)^4} + \frac{5!}{(2\pi n)^6} - \ldots, \quad (3·11)$$

and (3·10) gives the asymptotic expansions

$$K_{m,n}^{(0)} = \frac{1}{\pi^2}\left[\frac{n(\ln m - \ln n)}{(m^2 - n^2)} - \frac{1}{(2\pi m)^2}\left(\frac{1}{n} - \frac{3}{2\pi^2 n^3}\right) + \frac{6}{(2\pi m)^4 n} + \ldots\right] \quad (m \neq n), \quad (3·12)$$

$$K_{n,n}^{(0)} = \frac{1}{2\pi^2 n} - \frac{2}{\pi(2\pi n)^3} + \frac{24}{\pi(2\pi n)^5} - \ldots = \lim_{m \to n} [K_{m,n}^{(0)}] \quad (n \neq 0). \quad (3·13)$$

It also follows from (2·6) to (2·10), (3·4) and (3·11) that all inhomogeneous terms of (S_r) have asymptotic expansions of the same type:

$$b_m^{(r)} = B_0^{(r)} (2\pi m)^{-2} \ln m + B_1^{(r)} (2\pi m)^{-2} + B_2^{(r)} (2\pi m)^{-4} \ln m + B_3^{(r)} (2\pi m)^{-4} + \ldots, \quad (3·14)$$

where $B_0^{(r)}, \ldots, B_3^{(r)}, \ldots$ are numerical constants (some of which may vanish).

(a) *Asymptotic solution of the systems* (S_r)

If we compare (2·11) and (3·12) with I, (6·9), we see that, neglecting higher-order terms, $K_{m,n} = \tilde{K}_{m,n}$ $(m, n \geqslant 1)$. Thus, I, (6·9) is valid both for the wide and for the narrow double wedge; it may be shown that it remains valid in the intermediate case. Therefore, *the main term of* $K_{m,n}$ *in the asymptotic region (i.e. for* $\gamma_m \gg 1$, $\gamma_n \gg 1$*) is independent of* K.

This suggests that the behaviour of $K_{m,n}$ in the asymptotic region must be related to some feature of the field, the character of which remains unchanged for all K. The asymptotic behaviour of the evanescent mode amplitudes should also depend on this feature; this leads us to identify it with the *field singularity at the edges*.

Thus, we are led to infer that the asymptotic behaviour of $K_{m,n}$ as $\tilde{K}_{m,n}$ determines the asymptotic behaviour of the mode amplitudes. According to I, (4·18) and I, (5·2), we have, for a wide double wedge,

$$(-1)^n a_n \approx c_0(K)\, n^{-\frac{5}{3}} + c_1(K)\, n^{-\frac{7}{3}} + c_2(K)\, n^{-\frac{11}{3}} + O(n^{-\frac{13}{3}}). \tag{3·15}$$

It follows from the above assumption that an asymptotic expansion of the same type must be valid for all K. In particular, this should apply to the solution of all systems (S_r).

To find out whether this is correct, let us make the following asymptotic 'Ansatz' for the solution of (S_r):

$$a_n^{(r)} = A_0^{(r)} n^{-(1+\lambda_0)} + A_1^{(r)} n^{-(1+\lambda_1)} + A_2^{(r)} n^{-(1+\lambda_2)} + A_3^{(r)} n^{-(1+\lambda_3)} + \dots, \tag{3·16}$$

where $A_0^{(r)}, \dots, A_3^{(r)}, \dots$ are numerical constants, and $0 < \lambda_0 < \lambda_1 < \dots$. According to I, § 4 (*d*), the term in $n^{-\frac{5}{3}}$ in (3·15) gives rise to a singularity in $r^{-\frac{1}{3}}$ in the first partial derivatives of u, which corresponds to a term in $(kr)^{\frac{2}{3}}$ in the expansion of u near the edge. Similarly, each term of (3·16) gives rise to a singularity in some derivative of u, which corresponds to a given power of kr in the expansion of u near the edge. Integral powers of kr do not contribute, for they cannot give rise to a singularity; therefore, we may restrict ourselves to non-integral values of λ_i.

If we replace (3·16) in (S_r), the first member, $a_m^{(r)}$, is replaced by an asymptotic expansion in m. We may also replace $b_m^{(r)}$ by its asymptotic expansion (3·14). This suggests that we look for an asymptotic expansion of the remaining term, $\sum_{n=1}^{\infty} K_{m,n}^{(0)} a_n^{(r)}$, and then identify the results. This is the basis of the 'asymptotic method of solution'.

According to (3·12) and (3·16), what is required is the asymptotic expansion of the function

$$\Xi(\lambda, m) = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{(\ln m - \ln n)}{n^{\lambda}(m^2 - n^2)}, \tag{3·17}$$

for all positive non-integral values of λ. This expansion will be derived in the Appendix (§ 7). According to equations (7·1) and (7·22), the only term in the expansion which contains a non-integral power of m is: $\frac{1}{4}\sec^2(\frac{1}{2}\lambda\pi)\, m^{-\lambda-1}$. To each term $A_i^{(r)} m^{-\lambda_i-1}$ in the first member of (S_r) corresponds a term $\frac{1}{4}A_i^{(r)} \sec^2(\frac{1}{2}\lambda_i\pi)\, m^{-\lambda_i-1}$ in the second member. Since no other similar term exists, it follows, by identifying, that we must have

$$4\cos^2 \tfrac{1}{2}\lambda_i \pi = 1, \tag{3·18}$$

$$\lambda_{2j} = \tfrac{2}{3} + 2j; \quad \lambda_{2j+1} = \tfrac{4}{3} + 2j \quad (j = 0, 1, 2, \dots). \tag{3·19}$$

Equation (3·16) becomes

$$a_n^{(r)} = A_0^{(r)} n^{-\frac{5}{3}} + A_1^{(r)} n^{-\frac{7}{3}} + A_2^{(r)} n^{-\frac{11}{3}} + A_3^{(r)} n^{-\frac{13}{3}} + \ldots. \tag{3·20}$$

This agrees exactly with (3·15), justifying our previous assumptions.

There exists a close relationship between (3·20) and Sommerfeld's expansion of a branched wave function in a series of Bessel functions of fractional order (Frank & Von Mises 1935, p. 838). Equation (3·15) is the asymptotic form of Sommerfeld's expansion in the k-representation. According to (3·19) and (3·20), *the order of the branch points at the edges does not depend on* K. We shall call (3·20) a *Sommerfeld asymptotic series*.

To evaluate the coefficients of Sommerfeld's asymptotic series, we must substitute (3·20) in (S_r), and then, with the help of (7·1) and (7·22), we must identify the resulting asymptotic expansions. We shall outline the general procedure, and then we shall apply it to (S_0) and (S_4).

The asymptotic expansions (3·12), (3·14) and (7·22) are semi-convergent; for a given value of m, there exists, in each of them, an optimum number of terms, for which the error is a minimum. For a given number of terms, the accuracy increases with m; accurate results may be obtained even for $m = 1$, if we do not exceed the optimum number of terms corresponding to this case.

It may be expected that Sommerfeld's series will also be semi-convergent. However, we do not know whether the convergence is good already for $n = 1$. This depends on the numerical values of the coefficients $B_0^{(r)}$, $B_1^{(r)}$, ... in (3·14), so that a separate investigation is required for each system (S_r). For this purpose, we start with an 'Ansatz' containing the first N terms of Sommerfeld's series, assuming that it may be applied already for $n = 1$ (or $m = 1$). We replace this 'Ansatz' in (S_r), and replace the Ξ functions by their asymptotic expansions. Since our 'Ansatz' satisfies (3·18), all terms containing fractional powers of m drop out. The remaining terms are asymptotic expansions similar to (3·14). Since we have N constants at our disposal, we may identify the first N terms of these expansions, leaving a residual which contains only higher order terms. This gives a system of N linear equations to determine the unknowns $A_0^{(r)}, \ldots, A_{N-1}^{(r)}$.

This procedure may be repeated for $N = 1, 2, 3, \ldots$; in each approximation, we obtain the value of an additional unknown, as well as new values for the unknowns of the previous approximation. If the convergence is good, the values of a given unknown in different approximations should be close to one another; also, in each approximation, we must have: $|A_0^{(r)}| \gg |A_1^{(r)}| \gg |A_2^{(r)}| \gg \ldots$, for, otherwise, Sommerfeld's series would not converge well for $n = 1$, contrary to our assumption. If these conditions are satisfied for $N \leqslant N_0$, but not for $N = N_0 + 1$, we may take this as an indication that N_0 is the optimum number of terms for $n = 1$. In this way, we obtain an approximate solution of (S_r), in the form of a Sommerfeld series with N_0 terms.

It may happen, however, that Sommerfeld's series does not converge well for $n = 1$; or, if it does, we may want a better approximation. In either case, the following procedure may be applied: we assume that our 'Ansatz' of N terms applies only for $n \geqslant 2$, and deal with $a_1^{(r)}$ as an additional unknown. We separate (S_r) into two parts: the first equation $(m = 1)$, and the remaining system of equations $(m \geqslant 2)$. This partial system is dealt with in the same way as (S_r) in the previous case; thus, we obtain N equations in the $N+1$ unknowns

$a_1^{(r)}, A_0^{(r)}, \ldots, A_{N-1}^{(r)}$. Together with the additional equation for $m = 1$, this suffices to determine them. If Sommerfeld's series does not converge well for $n = 2$, we may start again, applying our 'Ansatz' only for $n \geqslant 3$ (this leaves two additional unknowns, $a_1^{(r)}$ and $a_2^{(r)}$), and so on. The applicability of the method is restricted only by the practical difficulty in solving large systems of linear equations, but we may expect, in general, that Sommerfeld's series converges well already for fairly small values of n.

The main result of the above discussion is that we may take advantage of our knowledge about the asymptotic behaviour of the solution of (S_r) to find an approximate solution, by reducing it to a finite system of linear equations.

(b) *Approximate solution of the narrow double-wedge problem*

Let us apply the method outlined in the previous section to (S_0) and (S_4). We shall take four terms in the 'Ansatz' (3·20); terms of higher order than n^{-4} (or m^{-4}) may then be neglected in (3·12), (3·14) and (7·22). The asymptotic method of solution leads to a system of four equations, the coefficients of which may be evaluated with the help of tables of Riemann's zeta function (see A). The result is

$$\left.\begin{aligned} -2{\cdot}45A_0^{(r)}+3{\cdot}60A_1^{(r)}+1{\cdot}28A_2^{(r)}+1{\cdot}15A_3^{(r)} &= -\tfrac{1}{4}B_0^{(r)}, \\ 8{\cdot}95A_0^{(r)}+8{\cdot}95A_1^{(r)}+0{\cdot}33A_2^{(r)}-0{\cdot}16A_3^{(r)} &= \tfrac{1}{4}B_1^{(r)}, \\ 0{\cdot}0405A_0^{(r)}+0{\cdot}153A_1^{(r)}+2{\cdot}45A_2^{(r)}-3{\cdot}60A_3^{(r)} &= B_2^{(r)}/(4\pi)^2, \\ 0{\cdot}0957A_0^{(r)}+0{\cdot}268A_1^{(r)}+8{\cdot}92A_2^{(r)}+8{\cdot}93A_3^{(r)} &= B_3^{(r)}/(4\pi)^2. \end{aligned}\right\} \tag{3·21}$$

To obtain the first approximation, $N = 1$, we take only the first equation and the first unknown in (3·21), neglecting the other equations and unknowns. In the second approximation, $N = 2$, we take the first two equations and unknowns, and so forth.

In the case of (S_0), we have, according to (2·5), (2·8), (3·4) and (3·14),

$$B_0^{(0)} = B_3^{(0)} = 8\mathrm{i}/\pi; \quad B_1^{(0)} = (8\mathrm{i}/\pi)\,[\ln(2\pi)+C]; \quad B_2^{(0)} = 0. \tag{3·22}$$

Substituting in (3·21), we obtain the successive approximations

$$\left.\begin{aligned} &N=1: \quad A_0^{(0)} = 0{\cdot}259\mathrm{i}; \\ &N=2: \quad A_0^{(0)} = 0{\cdot}207\mathrm{i}, \quad A_1^{(0)} = -0{\cdot}0353\mathrm{i}; \\ &N=3: \quad A_0^{(0)} = 0{\cdot}207\mathrm{i}, \quad A_1^{(0)} = -0{\cdot}0351\mathrm{i}, \quad A_2^{(0)} = -0{\cdot}00123\mathrm{i}; \\ &N=4: \quad A_0^{(0)} = 0{\cdot}207\mathrm{i}, \quad A_1^{(0)} = -0{\cdot}0354\mathrm{i}, \quad A_2^{(0)} = -0{\cdot}000108\mathrm{i}, \quad A_3^{(0)} = 0{\cdot}000747\mathrm{i}. \end{aligned}\right\} \tag{3·23}$$

The convergence is very good for $N \leqslant 3$, but begins to get worse for $N = 4$; this seems to indicate that the optimum number of terms for $n = 1$ is $N_0 = 3$. This leads to the following approximate solution of (S_0):

$$a_n^{(0)} = \mathrm{i}(0{\cdot}207n^{-\frac{5}{3}}-0{\cdot}0351n^{-\frac{7}{3}}-0{\cdot}00123n^{-\frac{11}{3}}) \quad (n \geqslant 1). \tag{3·24}$$

Substituting this in (3·8) and (3·4), we get

$$\frac{1}{\pi^2}\sum_{n=1}^{\infty}\tau_n\frac{a_n^{(0)}}{n} = 0{\cdot}061\mathrm{i}, \quad a_0^{(1)} = 2-1{\cdot}114\mathrm{i}, \tag{3·25}$$

$$B_0^{(4)} = B_3^{(4)} = -3{\cdot}242+1{\cdot}806\mathrm{i}, \quad B_1^{(4)} = -7{\cdot}829+1{\cdot}815\mathrm{i}, \quad B_2^{(4)} = 0. \tag{3·26}$$

Substituting (3·26) in (3·21), we find that the optimum number of terms for $n = 1$, in (S_4), is also $N_0 = 3$, which corresponds to the following approximation:

$$a_n^{(4)} = (-0{\cdot}263+0{\cdot}105\mathrm{i})\,n^{-\frac{5}{3}}+(0{\cdot}0447-0{\cdot}0543\mathrm{i})\,n^{-\frac{7}{3}}+(0{\cdot}00156+0{\cdot}00165\mathrm{i})\,n^{-\frac{11}{3}} \quad (n \geqslant 1). \tag{3·27}$$

Substituting these results in (3·6), (3·7) and (3·9), we finally get the *approximate solution of the narrow double-wedge problem*:

$$\begin{aligned} a_0 = {} & -1+(4\mathrm{i}/\pi)\,K\ln K+(2-1{\cdot}114\mathrm{i})\,K+(8/\pi^2)\,K^2(\ln K)^2-(4\mathrm{i}/\pi)\,(2-1{\cdot}114\mathrm{i})\,K^2\ln K \\ & -\tfrac{1}{2}(2-1{\cdot}114\mathrm{i})^2K^2-(16\mathrm{i}/\pi^3)\,K^3(\ln K)^3-(12/\pi^2)\,(2-1{\cdot}114\mathrm{i})\,K^3(\ln K)^2 \\ & +(4{\cdot}256+2{\cdot}377\mathrm{i})\,K^3\ln K+O(K^3), \end{aligned} \tag{3·28}$$

$$\begin{aligned} (-1)^n a_n = {} & (0{\cdot}207\,n^{-\frac{5}{3}}-0{\cdot}0351n^{-\frac{7}{3}}-0{\cdot}00123n^{-\frac{11}{3}})\,[\mathrm{i}K+(2/\pi)\,K^2\ln K-\tfrac{1}{2}\mathrm{i}\,(2-1{\cdot}114\mathrm{i})\,K^2 \\ & -(4\mathrm{i}/\pi^2)\,K^3(\ln K)^2]+[(-0{\cdot}263+0{\cdot}105\mathrm{i})\,n^{-\frac{5}{3}}+(0{\cdot}0447-0{\cdot}0543\mathrm{i})\,n^{-\frac{7}{3}} \\ & +(0{\cdot}00156+0{\cdot}00165\mathrm{i})\,n^{-\frac{11}{3}}]\,K^3\ln K+O(K^3) \quad (n \geqslant 1). \end{aligned} \tag{3·29}$$

What is the accuracy of this approximation? It is easily seen by the above derivation that the relative residuals (see I, § 5 (*b*)) of (3·28) and (3·29) are very small, even for $m = 1$, and decrease rapidly when m increases, tending to zero for $m \to \infty$. If we make the reasonable assumption that (S) is not an 'ill-conditioned' system (I, § 5 (*b*)), it follows that (3·28) and (3·29) are a very good approximation. Of course we must not go beyond the range of values of K for which the above series converge well, i.e. the above solution may be applied only if $|K\ln K| \ll 1$ (this means, in practice, $K \lesssim 0{\cdot}1$).

4. Other methods of solution

Although it is difficult, as we have seen, to estimate directly the accuracy of the asymptotic method of solution, an indirect estimate may be obtained by comparing it with other methods. We shall consider three different methods which may be applied to find an approximate solution of system (S).

(*a*) *The method of partial systems*

This method, which is due to Fourier, is often applied in problems of this type (Zernov 1951). It consists in approximating the solution of (S) by the solution of a sequence of partial systems extracted from it. In the Nth-order approximation, we neglect all equations and unknowns beyond the Nth, and solve the remaining system of N equations. If the method converges well, the values obtained for a given unknown in different approximations should be close to one another.

The first-order approximation is given by

$$a_0^{\mathrm{I}} = -K_{0,0}/(1-K_{0,0}); \quad a_n^{\mathrm{I}} = 0 \quad (n \geqslant 1). \tag{4·1}$$

Since this corresponds to neglecting all evanescent modes, we shall call the difference $\delta a_0 = a_0 - a_0^{\mathrm{I}}$ the *evanescent mode correction*. According to (3·28) and (4·1),

$$\begin{aligned} \delta a_0 = {} & 0{\cdot}061\mathrm{i}K[1-(4\mathrm{i}/\pi)\,K\ln K-(2-1{\cdot}145\mathrm{i})\,K-(12/\pi^2)\,K^2(\ln K)^2] \\ & -(0{\cdot}232-0{\cdot}088\mathrm{i})\,K^3\ln K+O(K^3). \end{aligned} \tag{4·2}$$

Therefore, $|\delta a_0| \ll |a_0|$, so that a_0^{I} is a very good approximation.

It is easily seen that a_0^{I} is identical to the result obtained by applying Levine & Schwinger's variational method, taking as 'trial function' a constant electric field over the aperture (Lewin 1951, p. 128; Marcuvitz 1951, p. 184). This is equivalent to neglecting the evanescent mode correction.

The second-order approximation, expressed in terms of δa_0 and a_1, gives:

$$\delta a_0^{\mathrm{II}} = 0{\cdot}041\mathrm{i}K[1-(4\mathrm{i}/\pi)\,K\ln K-(2-1{\cdot}154\mathrm{i})\,K-(12/\pi^2)\,K^2(\ln K)^2] \\ -(0{\cdot}157-0{\cdot}065\mathrm{i})\,K^3\ln K+O(K^3), \quad (4{\cdot}3)$$

$$a_1^{\mathrm{II}} = -0{\cdot}165[\mathrm{i}K+(2/\pi)\,K^2\ln K-(\mathrm{i}/2)\,(2-1{\cdot}134\mathrm{i})\,K^2-(4\mathrm{i}/\pi^2)\,K^3(\ln K)^2] \\ +(0{\cdot}210-0{\cdot}0515\mathrm{i})\,K^3\ln K+O(K^3), \quad (4{\cdot}4)$$

whereas, according to (3·29),

$$a_1 = -0{\cdot}171[\mathrm{i}K+(2/\pi)\,K^2\ln K-(\mathrm{i}/2)\,(2-1{\cdot}114\mathrm{i})\,K^2-(4\mathrm{i}/\pi^2)\,K^3(\ln K)^2] \\ +(0{\cdot}217-0{\cdot}0524\mathrm{i})\,K^3\ln K+O(K^3). \quad (4{\cdot}5)$$

Comparing (4·3) with (4·2), and (4·4) with (4·5), we are led to infer that the method of partial systems converges well. This is due to the rapid decrease of the amplitudes of the evanescent modes. The method of partial systems may be applied, in practice, to derive good approximations for the amplitudes of the reflected wave and of the first few evanescent modes, but not for higher-order modes, since the difficulty obviously increases with the number of modes which are retained.

(*b*) *Neumann's iteration method*

It is easily seen, by comparison with I, (6·16), that Neumann's series converges to the rigorous solution when $K = 0$. This leads us to expect that it still converges (very slowly) for $K \ll 1$ (I, § 6 (*b*)).

It follows from I, (6·7) that $a_0^{N\infty}$ will contain (among others) the following terms:

$$-K_{0,0}[1+K_{0,0}+(K_{0,0})^2+\ldots] = -K_{0,0}/(1-K_{0,0}) = a_0^{\mathrm{I}}, \quad (4{\cdot}6)$$

where the last equality follows from (4·1). Therefore, we may put

$$a_0^{N_r} = -K_{0,0}-(K_{0,0})^2-\ldots-(K_{0,0})^r+(\delta a_0)^{N_r}, \quad (4{\cdot}7)$$

where the last term represents the evanescent mode correction in Neumann's rth-order approximation.

According to (2·8), (3·11) and I, (6·4), we have, in N_1,

$$(\delta a_0)^{N_1} = 0, \quad (4{\cdot}8)$$

$$(-1)^n\,a_n^{N_1} = (0{\cdot}0779n^{-2}+0{\cdot}0323n^{-2}\ln n+0{\cdot}000816n^{-4}+\ldots)\mathrm{i}K \\ -0{\cdot}0323\mathrm{i}n^{-2}K^3\ln K+O(K^3) \quad (n \geqslant 1). \quad (4{\cdot}9)$$

Neumann's second-order approximation may be written

$$a_n^{N_2} = -K_{n,0}(1+K_{0,0})-(K'^2)_{n,0} \quad (n \geqslant 0), \quad (4{\cdot}10)$$

where

$$(K'^r)_{m,n} = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\ldots\sum_{v=1}^{\infty}K_{m,p}K_{p,q}\ldots K_{v,n} \quad (r \text{ factors}). \quad (4{\cdot}11)$$

It follows from (4·7), (4·10), (2·8), (2·9) and (3·11) that

$$(\delta a_0)^{N_2} = -(K'^2)_{0,0} = 0{\cdot}0136\mathrm{i}K - 0{\cdot}0102\mathrm{i}K^3 \ln K + O(K^3). \tag{4·12}$$

On the other hand, according to (2·8), (2·11), (3·12) and (4·11), for $n \geqslant 1$,

$$\begin{aligned}(-1)^n (\mathrm{i}\pi^3/K)\,(K'^2)_{n,0} = {} & -\Xi'_\lambda(1,n) + (2\pi)^{-2}[\zeta'(3) - (3/2\pi^2)\,\zeta'(5)]\,(\pi n)^{-2} \\ & + (2{\cdot}415 - K^2 \ln K)\,\{\Xi(1,n) - (2\pi)^{-2}\,[\zeta(3) - (3/2\pi^2)\,\zeta(5)]\,(\pi n)^{-2} + O(n^{-4})\} \\ & + (2\pi)^{-2}\{\Xi(3,n) - (2\pi)^{-2}[\zeta(5) - (3/2\pi^2)\,\zeta(7)]\,(\pi n)^{-2}\} + O(K^2),\end{aligned} \tag{4·13}$$

where $\Xi(\lambda, n)$ is the function defined in (3·17), and $\Xi'_\lambda(\lambda, n)$ is its partial derivative with respect to λ. The asymptotic expansion of these functions for $\lambda = 1$ is given in the Appendix (§ 7). With the help of the Appendix, we obtain, replacing (4·13) in (4·10), for $n \geqslant 1$,

$$\begin{aligned}(-1)^n a_n^{N_2} = n^{-2}\{ & [0{\cdot}124 + 0{\cdot}0556 \ln n + 0{\cdot}00395(\ln n)^2 + 0{\cdot}000545(\ln n)^3 \\ & - 0{\cdot}0000694\, n^{-2} \ln n + 0{\cdot}0000414\, n^{-2}\,(\ln n)^2 + O(n^{-2})]\,\mathrm{i}K + (0{\cdot}0390 + 0{\cdot}0162 \ln n \\ & + O(n^{-2}))\,[(2/\pi)\,K^2 \ln K - \tfrac{1}{2}\mathrm{i}(2 - 1{\cdot}175\mathrm{i})\,K^2] - [0{\cdot}0513 + 0{\cdot}00189 \ln n \\ & + 0{\cdot}00163\,(\ln n)^2 - 0{\cdot}000272\, n^{-2} \ln n + O(n^{-2})]\,\mathrm{i}K^3 \ln K\} + O(K^3).\end{aligned} \tag{4·14}$$

In N_3, we have

$$(\delta a_0)^{N_3} = -(1 + 2K_{0,0})\,(K'^2)_{0,0} - (K'^3)_{0,0}. \tag{4·15}$$

We find

$$(K'^3)_{0,0} = -0{\cdot}00148\mathrm{i}K + 0{\cdot}00104\mathrm{i}K^3 \ln K + O(K^3), \tag{4·16}$$

$$\begin{aligned}(\delta a_0)^{N_3} = {} & 0{\cdot}0272[\mathrm{i}K + (1/\pi)\,K^2 \ln K - \tfrac{1}{4}\mathrm{i}(2 - 1{\cdot}175\mathrm{i})\,K^2] - 0{\cdot}0204\mathrm{i}K^3 \ln K \\ & + 0{\cdot}00148\mathrm{i}K - 0{\cdot}00104\mathrm{i}K^3 \ln K + O(K^3).\end{aligned} \tag{4·17}$$

Let us compare these results with those of the asymptotic method. The series (4·6) converges (very slowly). Comparing (4·8), (4·12) and (4·17) with (4·2), we see that the evanescent mode correction also appears to converge very slowly in Neumann's method.

To compare (4·9) and (4·14) with (3·29), we replace the common factor $n^{\pm\frac{1}{3}}$, in the fractional powers of (3·29), by the expansion:

$$n^{\pm\frac{1}{3}} = \exp\,(\pm\tfrac{1}{3} \ln n) = 1 \pm \tfrac{1}{3} \ln n + \tfrac{1}{18}\,(\ln n)^2 + \ldots \tag{4·18}$$

(cf. I, (6·15)), with the following result:

$$\begin{aligned}(-1)^n a_n = {} & \{n^{-2}[0{\cdot}172 + 0{\cdot}0807 \ln n + 0{\cdot}00956\,(\ln n)^2 + 0{\cdot}00149\,(\ln n)^3 + \ldots] \\ & - n^{-4}[0{\cdot}00123 + 0{\cdot}00041 \ln n + 0{\cdot}00007\,(\ln n)^2 + \ldots]\}\,[\mathrm{i}K + (2/\pi)\,K^2 \ln K \\ & - \tfrac{1}{2}\mathrm{i}(2 - 1{\cdot}114\mathrm{i})\,K^2 - (4\mathrm{i}/\pi^2)\,K^3(\ln K)^2] + \{n^{-2}[(-0{\cdot}218 + 0{\cdot}051\mathrm{i}) \\ & + (-0{\cdot}103 + 0{\cdot}053\mathrm{i}) \ln n + (-0{\cdot}0121 + 0{\cdot}00283\mathrm{i})\,(\ln n)^2 + \ldots] + n^{-4}[(0{\cdot}00156 \\ & + 0{\cdot}00165\mathrm{i}) + (0{\cdot}00052 + 0{\cdot}00055\mathrm{i}) \ln n + \ldots] + \ldots\}\,K^3 \ln K + O(K^3).\end{aligned} \tag{4·19}$$

Comparing this with (4·9) and (4·14), we see that Neumann's method apparently converges also for $n \geqslant 1$. The rate of convergence, which is already slow for small n, becomes slower and slower as n increases, since we must then take a larger number of terms in (4·18) to get a good approximation. This behaviour in the asymptotic region is similar to that which was found in the wide double-wedge problem (I, § 6 (b)).

(c) *The modified iteration method*

The slow convergence of Neumann's method is due, in part, to the slow convergence of geometric series like (4·6). This suggests that the rate of convergence might be improved by separating the contribution of the principal mode from the others. System (S) may be rewritten as follows:

$$a_0 = -K_{0,0}(1-K_{0,0})^{-1} + (1-K_{0,0})^{-1} \sum_{n=1}^{\infty} K_{0,n} a_n = a_0^{\mathrm{I}} + \delta a_0, \tag{4·20}$$

$$a_m = -K_{m,0}(1-a_0) + \sum_{n=1}^{\infty} K_{m,n} a_n \quad (m \geqslant 1). \tag{4·21}$$

If we apply Neumann's method to the partial system (4·21), we get

$$a_m = -(1-a_0)\,[K_{m,0} + (K'^2)_{m,0} + (K'^3)_{m,0} + \ldots] \quad (m \geqslant 1). \tag{4·22}$$

Replacing (4·22) in (4·20), solving with respect to a_0, and replacing back in (4·22), we finally obtain

$$a_n = -[K_{n,0} + (K'^2)_{n,0} + \ldots]\,\{1 - [K_{0,0} + (K'^2)_{0,0} + \ldots]\}^{-1} \quad (n \geqslant 0). \tag{4·23}$$

It is easily seen by direct substitution that (4·23) (provided it converges) satisfies system (S).

The rth-order approximation of the 'modified iteration method' is obtained by taking the first r terms of (4·22):

$$a_0^{M_r} = -[K_{0,0} + \ldots + (K'^{r+1})_{0,0}]\,\{1 - [K_{0,0} + \ldots + (K'^{r+1})_{0,0}]\}^{-1}, \tag{4·24}$$

$$a_n^{M_r} = -[K_{n,0} + \ldots + (K'^{r})_{n,0}]\,\{1 - [K_{0,0} + \ldots + (K'^{r+1})_{0,0}]\}^{-1} \quad (n \geqslant 1). \tag{4·25}$$

The first- and second-order approximations may be evaluated without difficulty, with the help of (4·12), (4·13) and (4·16). The results are

$$\begin{aligned}(\delta a_0)^{M_1} = {} & 0{\cdot}0544\mathrm{i}K[1 - (4\mathrm{i}/\pi)\,K \ln K - (2 - 1{\cdot}148\mathrm{i})\,K - (12/\pi^2)\,K^2(\ln K)^2] \\ & - (0{\cdot}208 - 0{\cdot}079\mathrm{i})\,K^3 \ln K + O(K^3),\end{aligned} \tag{4·26}$$

$$\begin{aligned}(-1)^n a_n^{M_1} = {} & [n^{-2}(0{\cdot}156 + 0{\cdot}0646 \ln n) + O(n^{-4})]\,[\mathrm{i}K + (2/\pi)\,K^2 \ln K \\ & - (\mathrm{i}/2)\,(2 - 1{\cdot}121\mathrm{i})\,K^2 - (4\mathrm{i}/\pi^2)\,K^3\,(\ln K)^2] + \{n^{-2}[(-0{\cdot}099 - 0{\cdot}009\mathrm{i}) \\ & + (-0{\cdot}041 + 0{\cdot}023\mathrm{i}) \ln n] + O(n^{-4})\}\,K^3 \ln K + O(K^3) \quad (n \geqslant 1),\end{aligned} \tag{4·27}$$

$$\begin{aligned}(\delta a_0)^{M_2} = {} & 0{\cdot}0604\mathrm{i}K[1 - (4\mathrm{i}/\pi)\,K \ln K - (2 - 1{\cdot}145\mathrm{i})\,K - (12/\pi^2)\,K^2(\ln K)^2] \\ & - (0{\cdot}231 - 0{\cdot}087\mathrm{i})\,K^3 \ln K + O(K^3),\end{aligned} \tag{4·28}$$

$$\begin{aligned}(-1)^n a_n^{M_2} = {} & \{n^{-2}[0{\cdot}170 + 0{\cdot}0788 \ln n + 0{\cdot}00790\,(\ln n)^2 + 0{\cdot}00109\,(\ln n)^3] \\ & - n^{-4}[0{\cdot}00014 \ln n - 0{\cdot}000083\,(\ln n)^2] + O(n^{-4})\}\,[\mathrm{i}K + (2/\pi)\,K^2 \ln K \\ & - (\mathrm{i}/2)\,(2 - 1{\cdot}115\mathrm{i})\,K^2 - (4\mathrm{i}/\pi^2)\,K^3\,(\ln K)^2] + \{n^{-2}[(-0{\cdot}216 + 0{\cdot}051\mathrm{i}) \\ & + (-0{\cdot}100 + 0{\cdot}052\mathrm{i}) \ln n + (-0{\cdot}0101 + 0{\cdot}0023\mathrm{i})\,(\ln n)^2] + (0{\cdot}00018 + 0{\cdot}00044\mathrm{i}) \\ & \times n^{-4} \ln n + O(n^{-4})\}\,K^3 \ln K + O(K^3) \quad (n \geqslant 1).\end{aligned} \tag{4·29}$$

Comparing these results with (4·2) and (4·19), we see that the modified iteration method apparently converges much more rapidly than Neumann's method. Expression (4·28) almost does not differ from (4·2). Expression (4·29) agrees with (3·29) within 5 % for $n \lesssim 10$, and within 15 % for $n \lesssim 10^2$; the convergence becomes worse as n increases. Therefore, the second-order approximation of the modified iteration method is a very good approximation, except at extremely small distances from the edges.

The modified iteration method seems to be the best of the three methods studied in this section. The agreement between the results obtained by these methods and the solution by the asymptotic method justifies the assumption that (S) is not an 'ill-conditioned' system and indicates that (3·28) and (3·29) give a very good approximation of the rigorous solution.

5. Applications

In this section we shall apply (3·28) and (3·29) to the evaluation of physically important quantities.

(a) *The reflexion coefficient*

It follows from (1·2) and (3·28) that

$$|R|^2 = 1-4K+8K^2+(16/\pi^2)\,K^3(\ln K)^2-2{\cdot}84K^3\ln K+O(K^3)$$
$$= 1-12{\cdot}57q+78{\cdot}96q^2+50{\cdot}27q^3[\ln(\pi q)]^2-88{\cdot}03q^3\ln(\pi q)+O(q^3), \qquad (5{\cdot}1)$$

where $q = K/\pi = d/\lambda$ ($d = 2a$ = width of double wedge).

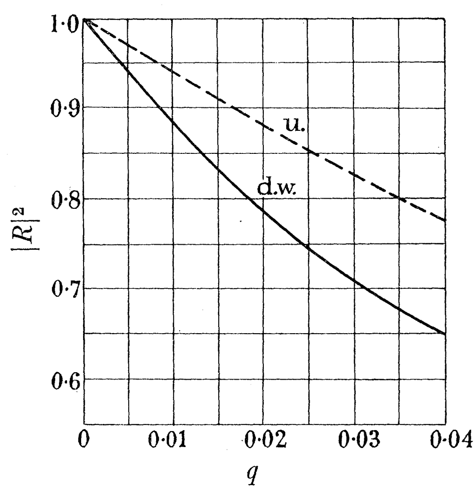

Figure 1. Reflexion coefficient as a function of q. d.w., double wedge; u. unflanged parallel-plate waveguide.

The curve in full line in figure 1 represents $|R|^2$ as a function of q. The curve in dashed line gives the reflexion coefficient of an unflanged semi-infinite parallel-plate waveguide, which is given by (Vajnshtejn 1954, p. 22):

$$|R|^2 = \exp(-2\pi q) \quad (\text{for } 0 < q < 1). \qquad (5{\cdot}2)$$

The presence of a flange results in a better matching to outside space, and therefore reduces the reflexion coefficient.

Any deviation of the reflexion coefficient from its limiting value unity may be called a 'radiative correction'. There are two kinds of contributions to the radiative corrections. One of them is due to the principal mode by itself. The surface current associated with this mode travels along the waveguide walls (in the x-direction), then 'turns the corner' upon its arrival at the edges, and propagates in the y-direction on the screen, bringing sources of radiation into region II. The other contribution is due to the evanescent mode correction (4·2). The evanescent modes do not contribute directly to the energy flux; they serve as a storage of energy to build up the quasi-static fields attached to the edges. However, they contribute to the radiative corrections through their coupling with the principal mode (see (3·8) and (3·9)). The evanescent mode correction is usually neglected in approximate treatments of radiation from waveguides. Our solution indicates that it is very small for a narrow double wedge. This is due to the weak coupling between the principal mode and the evanescent modes (I, § 5 (*g*)).

(*b*) *The end correction*

Not only the modulus, but also the phase of a_0 play an important role. If we put

$$a_0 = -|R| \exp(\mathrm{i}\Phi), \tag{5·3}$$

(1·1) becomes

$$u_{\mathrm{I}}(x,y) = (1-|R|)\exp(-\mathrm{i}kx) - 2|R|\exp[\tfrac{1}{2}\mathrm{i}(\pi+\Phi)]\sin[k(x+\alpha)] + \sum_{n=1}^{\infty}(\ldots), \tag{5·4}$$

where

$$\alpha = \Phi/2k. \tag{5·5}$$

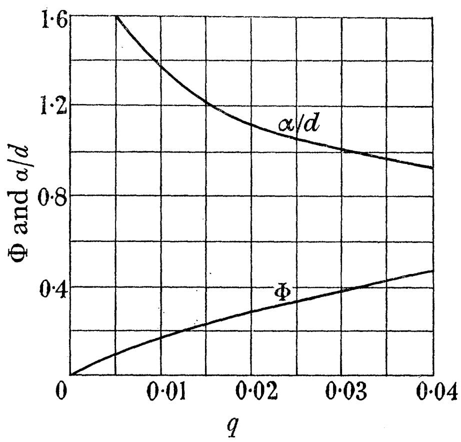

Figure 2. Phase of reflected wave and ratio of end correction to diameter as a function of q.

Thus, interference between the incident mode and the reflected mode gives rise to *quasi-stationary waves* within the waveguide. The distribution of nodes and anti-nodes of the stationary part of the field is determined by the parameter α defined in (5·5). The correct distribution is obtained by taking the first (virtual) node at $x = -\alpha$; hence, the name of 'end correction'. According to (3·28), (5·3) and (5·5),

$$\Phi = -(4/\pi)K\ln K + 1{\cdot}114K + (16/3\pi^3)K^3(\ln K)^3 - (4\times 1{\cdot}114/\pi^2)K^3(\ln K)^2 - 0{\cdot}621\,K^3\ln K + O(K^3), \tag{5·6}$$

$$\pi\alpha/d = -\ln(\pi q) + 0{\cdot}875 + \tfrac{4}{3}q^2[\ln(\pi q)]^3 - 3{\cdot}50q^2[\ln(\pi q)]^2 - 4{\cdot}81\,q^2\ln(\pi q) + O(q^2). \tag{5·7}$$

Figure 2 is a plot of Φ and α/d as a function of q. The ratio α/d tends to infinity for $q \to 0$, but the behaviour of the end correction depends on whether $d \to 0$ or $\lambda \to \infty$; it tends to zero in the first case, and to infinity in the second.

The double-wedge problem also admits an acoustical interpretation (I, § 2). The acoustical problem was discussed by Rayleigh (1904), with the assumption of a constant normal velocity over the aperture. This is obviously equivalent to neglecting all evanescent modes (cf. § 4 (*a*)). Rayleigh found the following value for the end correction (adapted to our notation), for $K \ll 1$:

$$\pi\alpha/d = -\ln(\pi q) + \tfrac{3}{2} - C \quad (\tfrac{3}{2} - C \cong 0{\cdot}923). \tag{5·8}$$

It is easily seen that this follows from (5·7), by neglecting the evanescent mode correction. The contribution of the evanescent mode correction is of the order of 2 % for $q = 0{\cdot}05$.

(*c*) *Resonance of a finite channel*

The reflexion coefficient and the end correction play an important role in the analysis of resonance effects in channels of finite length. As an example, let us consider the system shown in figure 3, where the waveguide is stopped at a distance L from the mouth by a perfectly reflecting piston P. Let this system be excited by a plane wave, perpendicularly incident from region II, and let us assume not only that $K \ll 1$, but also that $a \ll L$. As a result of the last assumption, evanescent modes excited at one extremity will practically disappear before reaching the other one, so that the analysis may be confined to the principal mode.

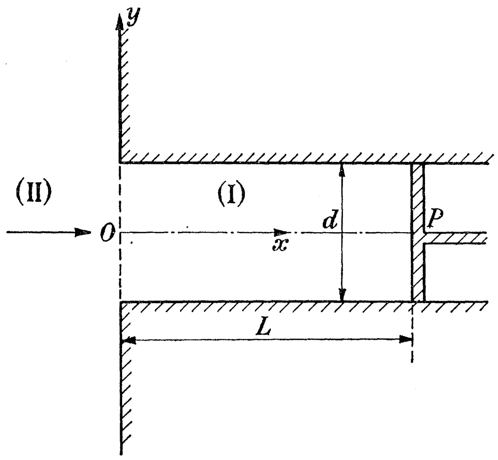

FIGURE 3. Channel of finite length closed at one end.

The principal mode is excited with amplitude A at the open end, travels to the piston, is totally reflected, travels back to the open end, is again reflected, and so on. According to our assumptions, the amplitude and phase of the waves reflected at the open end may be calculated by means of (5·1) and (5·6). Taking into account multiple reflexion, and neglecting evanescent modes, we find the following expression for the wave function in region I:*

$$u_{\mathrm{I}}(x) = 2A \exp(\mathrm{i}kL)\,\{1 + |R| \exp[\mathrm{i}(\Phi + 2kL)]\}^{-1} \cos[k(x-L)]. \tag{5·9}$$

Thus, stationary waves build up within the channel, with an intensity proportional to

$$g^2 = |1 + |R| \exp[\mathrm{i}(\Phi + 2kL)]|^{-2} = |1 + 2|R| \cos[4\pi(L+\alpha)/\lambda] + |R|^2|^{-1}. \tag{5·10}$$

For given d and λ, the intensity is a maximum for the resonant lengths

$$L_n + \alpha = (2n+1)\,\tfrac{1}{4}\lambda \quad (n = 0, 1, 2, \ldots). \tag{5·11}$$

This agrees with the well-known result of the elementary theory of organ pipes, provided that this result is applied to the 'corrected length', i.e. to the sum of the actual length of the pipe and the end correction.

* This treatment follows closely that of Vajnshtejn (1954, p. 111).

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If L is changed, by displacing piston P, we find, in the neighbourhood of the nth resonance,

$$g^2 = |1-2|R|\cos[4\pi(L-L_n)/\lambda]+|R|^2|^{-1}. \tag{5·12}$$

For $|L-L_n| \ll \lambda/4\pi$, this has the form of a typical resonance curve:

$$[g(w)/g_{\text{max.}}]^2 = (\Delta w)^2[4w^2+(\Delta w)^2]^{-1}, \tag{5·13}$$

where

$$w = (L-L_n)/d, \tag{5·14}$$

$$g^2_{\text{max.}} = (1-|R|)^{-2} \cong (2\pi q)^{-2}, \tag{5·15}$$

$$\Delta w = (1-|R|)/(2\pi q\sqrt{|R|}) \cong |R|^{-\frac{1}{2}} \text{ ('half-width')}. \tag{5·16}$$

By measuring the resonant length and half-width of a resonance, the reflexion coefficient and the end correction might be determined. On the other hand, the values of these quantities might be employed to measure the wavelength.

The effect of a finite conductivity of the walls may be taken into account, by applying well-known methods (Slater 1942). It may be shown that, for good conductors, in the microwave region, the correction due to finite conductivity may be rendered much smaller than the evanescent mode correction, so that the latter might be detected, in principle, by a very precise measurement.

A similar treatment may be applied to the resonance of a channel open at both ends. Such a system may be considered as one of the simplest theoretical models of an antenna.

(d) *The radiation pattern*

To find the radiation pattern of the narrow double wedge, we must evaluate the coefficients $A(\gamma)$, which, according to I, (5·22) to (5·24), are given by

$$A(\gamma) = (2/\pi)(1-\gamma^2)^{-\frac{1}{2}}\gamma\sin(\gamma K)\left[(1-a_0)\gamma^{-2}+\mathrm{i}\sum_{n=1}^{\infty}(-1)^n(\gamma_n^2-1)^{\frac{1}{2}}(\gamma_n^2-\gamma^2)^{-1}a_n\right]. \tag{5·17}$$

If we replace $(-1)^n a_n$ by (3·29), and expand $(\gamma_n^2-1)^{\frac{1}{2}}$ in powers of γ_n^{-1}, the series which appears in (5·17) may be expressed in terms of the functions

$$\mathscr{Y}(\lambda,\chi) = \sum_{n=1}^{\infty} n^{-\lambda}(n^2-\chi^2)^{-1}, \tag{5·18}$$

where $\chi = \gamma K/\pi = \gamma/\gamma_1$, and λ takes the values $\frac{2}{3}, \frac{4}{3}, \ldots$. For $\chi \ll 1$, $\mathscr{Y}(\lambda,\chi)$ may be expanded in a power series in χ (see A), and (5·17) becomes

$$A(\gamma) = (2/\pi)(1-\gamma^2)^{-\frac{1}{2}}\gamma\sin(\gamma K)\{(1-a_0)\gamma^{-2}-(0{\cdot}223/\pi)K^2[1-(2\mathrm{i}/\pi)K\ln K]+\ldots\}$$
$$(\gamma \ll \gamma_1) \tag{5·19}$$

Substituting this result in I, (5·32), we find

$$u_{\text{II}}(R,\theta) \approx [(1-a_0)\sin(K\sin\theta)/\sin\theta-(0{\cdot}071\,K^2-0{\cdot}045\mathrm{i}K^3\ln K)$$
$$\times\sin\theta\sin(K\sin\theta)+\ldots]\,H_0(kR). \tag{5·20}$$

The first term in the second member, which is the main term, differs from Kirchhoff's approximation I, (5·33) only by the replacement of a factor equal to unity by $(1-a_0)$. This is equivalent to the replacement of the incident wave by the sum of the incident and reflected waves in Kirchhoff's aperture distribution.

It follows from (5·20) that

$$u_{\mathrm{II}}(R,\theta) \approx [(1-a_0)\,K + O(K^3 \sin^2\theta)]\,H_0(kR), \tag{5·21}$$

so that the main term of the radiation field is an isotropic cylindrical wave. At large distances, the aperture behaves like a virtual line source.

The mean outgoing energy flux (per unit length in the z-direction) is

$$\Phi_{\mathrm{II}} = \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \sigma(\theta)\,\mathrm{d}\theta, \tag{5·22}$$

where $\sigma(\theta)$ has been defined in I, (5·40). The conservation of energy demands that

$$\Phi_{\mathrm{II}} = \Phi_{\mathrm{I}} = ca(1-|R|^2)/4\pi. \tag{5·23}$$

This condition, which is easily verified with the help of (5·1), is an important test of the accuracy of the solution.

For $\chi \gtrsim 1$, we may obtain an asymptotic expansion of $\mathscr{Y}(\lambda,\chi)$ and of $A(\gamma)$ by a method similar to that which was applied to the function $\Xi(\lambda, m)$ (see Appendix). For $\gamma \gg \gamma_1$, we find (see A)

$$A(\gamma) \approx -2(\pi/K)^{\frac{2}{3}}\,0{\cdot}207\mathrm{i}K[1-(2\mathrm{i}/\pi)\,K\ln K + \ldots]\cos(\gamma K - \tfrac{1}{3}\pi)\,\gamma^{-\frac{5}{3}}. \tag{5·24}$$

This gives the correct singularity at the edges.

For distances from the edges much smaller than a, the field has a quasi-static behaviour, determined by the singularity. For distances much larger than the wavelength, the radiation field (5·20) predominates. The evaluation of the field in the intermediate region, where the transition from the quasi-static zone to the wave zone takes place, is more difficult, and will not be considered here.

6. Summary and conclusions

The main interest of the double-wedge problem is that it seems to be the simplest diffraction problem, involving one finite dimension, which may be treated in the k-representation.

The rigorous formulation of the problem reduced it to the solution of an infinite system of linear equations. Although this appears to involve considerable difficulties, approximate solutions may be found in a fairly direct way, once we have recognized the essential physical aspects of the problem. A drawback of this method is the lack of a rigorous procedure for estimating the error. However, we believe that the approximations given in this paper are well justified, from the physical point of view.

To find approximate solutions, we had to determine the main limiting cases of our problem. It is usually assumed that there are only two such cases: the short-wavelength limit and the long-wavelength limit. Perhaps the most significant result of our analysis is that there exists a third limiting case, the case of critical incidence, which plays an important role as a link between the theory of diffraction and the theory of quasi-stationary currents. This is an essential difference between diffraction problems involving no finite dimension and those which involve at least one finite dimension, and is at the root of some of the main difficulties encountered in the latter case.

The wide double-wedge problem may be treated as a perturbation of a limiting case, in which $K \to \infty$. The most convenient limiting process seems to be the removal of one wedge to infinity. In this process, the solution of the double-wedge problem tends almost everywhere

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(not everywhere, because of the critical region) to the corresponding solution of Reiche's single-wedge problem. The limiting form of system (S) is an integral equation, which is satisfied by the solution of the single-wedge problem in the k-representation. Reiche's solution allowed us to determine the behaviour of the field near the edges. This led to the replacement of the 'edge condition' by a condition of great heuristic value, on the asymptotic behaviour of the mode amplitudes.

Treating the wide double-wedge problem, for ordinary incidence, as a perturbation of the single-wedge problem, we obtained the independent wedges approximation. The i.w.a. is related to the first-order approximation in Schwarzschild's theory of diffraction by a slit, but it contains certain improvements. In terms of Schwarzschild's picture of interaction between the wedges by multiple diffraction, the i.w.a. is a 'weak coupling' approximation. The coupling affects most strongly modes which belong to the critical region. Outside of this region, the i.w.a. must be a very good approximation, for ordinary incidence; this is confirmed by the evaluation of the residuals. In the intermediate wavelength region, the i.w.a. would probably need to be improved. Although we have not investigated the problem in this region, we believe that this could be done by methods similar to those which were recently developed in connexion with other problems (Karp & Russek 1956; Keller 1957).

For ordinary incidence, the i.w.a. allows us to throw some light on questions A and B of I, §1. It has been shown in a previous paper (Nussenzveig 1959*a*) that all classical diffraction patterns depend only on a very small spectral region, centred on the direction of incidence. The i.w.a. shows that, for ordinary incidence, Kirchhoff's approximation is the main term of the solution in this region, though this is by no means true outside of it. This result may be justified by the closeness to geometrical optics. Taken together with the previous result, it explains the success of classical diffraction theory.

The difficulties for understanding this in the x-representation arise from an inadequate choice of variables: attention is focused on the field distribution in the plane of the screen, while the proper variables are the Fourier components of this distribution, belonging to a narrow spectral region. Any details which do not contribute significantly to this region (such as edge singularities) do not affect the classical patterns. Their effect may be felt only in regions which lie outside the domain of classical theory, e.g. at large diffraction angles, or at small distances from the aperture.

The error of classical diffraction theory is of the same order as the reflexion coefficient, which, for ordinary incidence, is extremely small in the visible domain. The origin of reflexion is the distortion of the incident mode, which takes place chiefly within a few wavelengths from the edges. The distortion is much larger in the critical incidence region, where the i.w.a. breaks down, and strong reflexion appears.

In the limiting case of exactly critical incidence, the rigorous solution may be given for any value of the wavelength. The incident mode is totally reflected, and the field vanishes everywhere, so that no propagation is possible. The critical mode is not coupled to the others, as shown by the 'separation' of system (S). In the neighbourhood of exactly critical incidence the coupling to other modes is weak; the effects which appear in this region are illustrated by the solution of the narrow double-wedge problem.

To investigate the possibility that Kirchhoff's approximation is the first step of an accurate solution by successive approximations (I, §1, question C), Neumann's iteration method was applied to (S). By comparing Neumann's first- and second-order approximations, for ordinary incidence, with the i.w.a., we have seen that the numerical convergence of Neumann's method appears to be satisfactory in the normal region, but not so in the asymptotic region. It follows that Neumann's method may be applied, in practice, to the evaluation of quantities which do not depend on the asymptotic region (such as the reflexion coefficient).

For exactly critical incidence, Neumann's series converges to the rigorous solution, although this is a case of 'strong coupling' between the wedges; the convergence, however, is very slow. The results obtained in the narrow double-wedge problem seem to indicate that the iteration method still converges (very slowly) in the neighbourhood of exactly critical incidence. This suggests that Neumann's method converges for all values of the wavelength, but the rate of convergence varies greatly from one region to another.

In the narrow double-wedge problem, the effects of strong reflexion, typical of critical incidence, and quasi-static effects due to the edge zones, predominate. Since both effects exist also for a wide double wedge, the connexion between short- and long-wavelength regions (I, § 1, question D) becomes quite clear in this case. This connexion suggested the 'asymptotic method of solution'. This method is based on the determination of the asymptotic behaviour of the mode amplitudes (given by 'Sommerfeld's asymptotic series'). Advantage is taken of this to reduce each system (S_r), resulting from the separation of (S), to a finite system of linear equations. In this way, we obtain an approximate solution of the narrow double-wedge problem, which seems to be a very good approximation. This is confirmed by its agreement with approximate solutions derived by different methods. The best of these seems to be the 'modified iteration method'.

The solution obtained by the asymptotic method allowed us to evaluate the 'radiative corrections' to the amplitude and phase of the reflected wave. The main correction arises from the principal mode by itself, but there is an additional contribution, due to its (weak) coupling with the evanescent modes. This 'evanescent mode correction' is very small for a narrow double wedge; it was neglected in previous treatments of this problem by Rayleigh and by Levine & Schwinger's variational method. The reflexion coefficient and the end correction may be employed in the analysis of resonance effects in channels of finite length, a subject of some interest in connexion with antenna theory.

To conclude, a few possible applications and topics for further developments will be mentioned; some of them are now being investigated. A study of the solution in the intermediate wavelength region would be desirable, in connexion with the following problems: the possibility of extending the asymptotic method of solution to this region; the behaviour of the radiative corrections in the case of strong coupling between the modes; the evolution of the radiation pattern in the transition from the long-wavelength region to the domain of classical diffraction theory; the problem of critical incidence of higher-order modes. A subject which we have barely touched upon, and which deserves further consideration, is the problem of border effects in optics. Heisenberg's uncertainty relations have often been illustrated by means of diffraction experiments, but the analysis of these problems stands in need of some improvement.* Finally, it is hoped that the methods developed in this paper may help to throw some light on the antenna problem.

* See Beck & Nussenzveig 1958 *Nuovo Cim.* **9**, 1068.

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7. Appendix. Asymptotic expansion of $\Xi(\lambda, m)$

The function $\Xi(\lambda, m)$ defined in (3·17) satisfies the recurrence relation

$$\Xi(\lambda+2, m) = m^{-2}\Xi(\lambda, m) + \pi^{-2}\zeta(\lambda+2)\, m^{-2}\ln m + \pi^{-2}\zeta'(\lambda+2)\, m^{-2}, \tag{7·1}$$

where $\zeta(s)$ is Riemann's zeta function, and $\zeta'(s) = \mathrm{d}\zeta(s)/\mathrm{d}s$. It is easily seen that

$$\Xi(\lambda, m) = \pi^{-2}[\tfrac{1}{2}m^{-\lambda-2} + \xi(\lambda, m)\ln m + \xi'_\lambda(\lambda, m)], \tag{7·2}$$

where
$$\xi(\lambda, m) = \sum_{n=1}^{\infty}{}' \, n^{-\lambda}(m^2-n^2)^{-1}, \tag{7·3}$$

the prime on the summation sign indicating the exclusion of $n = m$.

We shall restrict ourselves, at first, to non-integral values of λ. According to (7·1), it suffices to consider the interval $0 < \lambda < 2$. To find the asymptotic expansion of $\xi(\lambda, m)$ in this interval, we start from the identity (Valiron 1948, p. 507)

$$n^{-\lambda} = (2/\pi)\sin\tfrac{1}{2}\lambda\pi \int_0^\infty (x^2+n^2)^{-1}\, x^{1-\lambda}\,\mathrm{d}x \quad (0<\lambda<2). \tag{7·4}$$

Substituting this in (7·3), and taking into account the following results,

$$\sum_{n=1}^{\infty}{}' (m^2-n^2)^{-1} = -\tfrac{3}{4}m^{-2} \quad \text{(Bromwich 1949, p. 67)}, \tag{7·5}$$

$$\sum_{n=1}^{\infty} (x^2+n^2)^{-1} = (\pi/2x)\,\mathscr{L}(\pi x) \quad \text{(Valiron, p. 507)}, \tag{7·6}$$

where $\mathscr{L}(u)$ is the Langevin function, $\mathscr{L}(u) = \coth u - u^{-1}$, we finally obtain

$$\xi(\lambda, m) = -\tfrac{1}{4}(2\lambda+3)\, m^{-\lambda-2} + X(\lambda, m), \tag{7·7}$$

where
$$X(\lambda, m) = \sin\tfrac{1}{2}\lambda\pi \int_0^\infty (x^2+m^2)^{-1}\,\mathscr{L}(\pi x)\, x^{-\lambda}\,\mathrm{d}x. \tag{7·8}$$

For $0<\lambda<2$, we have

$$X(\lambda, m) = \tfrac{1}{2}\pi\tan(\tfrac{1}{2}\lambda\pi)\, m^{-\lambda-1} + \tfrac{1}{2}m^{-\lambda-2} + \zeta(\lambda)\, m^{-2} - 2\sin(\tfrac{1}{2}\lambda\pi)\, m^{-2}\eta(\lambda, m), \tag{7·9}$$

where
$$\eta(\lambda, m) = \int_0^\infty (\mathrm{e}^{2\pi x}-1)^{-1}\,(x^2+m^2)^{-1}\, x^{2-\lambda}\,\mathrm{d}x. \tag{7·10}$$

For $1<\lambda<2$, (7·9) follows from the formula (Valiron, p. 507)

$$\zeta(\lambda) = \sin\tfrac{1}{2}\lambda\pi \int_0^\infty \mathscr{L}(\pi x)\, x^{-\lambda}\,\mathrm{d}x \quad (1<\lambda<2), \tag{7·11}$$

with the help of (7·4). For $0<\lambda<1$, (7·9) follows from the formula (Titchmarsh 1951, p. 25)

$$\zeta(\lambda) = \sin\tfrac{1}{2}\lambda\pi \int_0^\infty \left[\frac{2}{\mathrm{e}^{2\pi x}-1} - \frac{1}{\pi x}\right] x^{-\lambda}\,\mathrm{d}x \quad (0<\lambda<1). \tag{7·12}$$

Substituting (7·9) in (7·7), we find (for $0 < \lambda < 2$)

$$\xi(\lambda, m) = \tfrac{1}{2}\pi \tan(\tfrac{1}{2}\lambda\pi)\, m^{-\lambda-1} - \tfrac{1}{4}(2\lambda+1)\, m^{-\lambda-2} + \zeta(\lambda)\, m^{-2} - 2m^{-2} \sin(\tfrac{1}{2}\lambda\pi)\, \eta(\lambda, m). \tag{7·13}$$

To find the asymptotic expansion of $\eta(\lambda, m)$, we employ the identity

$$(x^2+m^2)^{-1} = m^{-2} \sum_{r=0}^{N-1} (-1)^r (x/m)^{2r} + (-1)^N (x^2+m^2)^{-1} (x/m)^{2N}. \tag{7·14}$$

Substituting this in (7·10), and employing Riemann's formula (Valiron, p. 506)

$$\zeta(s)\, \Gamma(s) = \int_0^\infty \frac{t^{s-1}\, dt}{e^t - 1} \quad (s > 1), \tag{7·15}$$

we find

$$\eta(\lambda, m) = (2\pi)^{\lambda-1} \sum_{n=1}^{N} (-1)^{n-1} \zeta(2n+1-\lambda)\, \Gamma(2n+1-\lambda)\, (2\pi m)^{-2n} + \epsilon(\lambda, m), \tag{7·16}$$

where

$$\epsilon(\lambda, m) = (-1)^N m^{-2N} \int_0^\infty \frac{x^{2N+2-\lambda}\, dx}{(e^{2\pi x}-1)(x^2+m^2)},$$

$$|\epsilon(\lambda, m)| < (2\pi)^{\lambda-1} \zeta(2N+3-\lambda)\, \Gamma(2N+3-\lambda)\, (2\pi m)^{-2N-2}. \tag{7·17}$$

Equation (7·16) gives the asymptotic expansion of $\eta(\lambda, m)$ for $0 < \lambda < 2$. According to (7·17), the remainder after a certain number of terms is of the same sign as, and numerically less than, the first term which is neglected.

Differentiating (7·13) with respect to λ, we find

$$\begin{aligned}\xi'_\lambda(\lambda, m) = & -\tfrac{1}{2}\pi \tan(\tfrac{1}{2}\lambda\pi)\, m^{-\lambda-1} \ln m + \tfrac{1}{4}\pi^2 \sec^2(\tfrac{1}{2}\lambda\pi)\, m^{-\lambda-1} \\ & + \tfrac{1}{4}(2\lambda+1)\, m^{-\lambda-2} \ln m - \tfrac{1}{2} m^{-\lambda-2} + \zeta'(\lambda)\, m^{-2} - \pi m^{-2} \cos(\tfrac{1}{2}\lambda\pi)\, \eta(\lambda, m) \\ & - 2m^{-2} \sin(\tfrac{1}{2}\lambda\pi)\, \eta'_\lambda(\lambda, m).\end{aligned} \tag{7·18}$$

According to (7·16),

$$\begin{aligned}\eta'_\lambda(\lambda, m) = \ln(2\pi)\, \eta(\lambda, m) + (2\pi)^{\lambda-1} \sum_{n=1}^{N} (-1)^n \left[\frac{\zeta'(2n+1-\lambda)}{\zeta(2n+1-\lambda)} + \frac{\Gamma'(2n+1-\lambda)}{\Gamma(2n+1-\lambda)}\right] \\ \times \zeta(2n+1-\lambda)\, \Gamma(2n+1-\lambda)\, (2\pi m)^{-2n} + \epsilon'_\lambda(\lambda, m),\end{aligned} \tag{7·19}$$

where

$$\epsilon'_\lambda(\lambda, m) = (-1)^{N+1} m^{-2N} \int_0^\infty \frac{x^{2N+2-\lambda} \ln x\, dx}{(e^{2\pi x}-1)(x^2+m^2)}, \tag{7·20}$$

$$\begin{aligned}|\epsilon'_\lambda(\lambda, m)| < (2\pi)^\lambda (2\pi m)^{-2N-2} [\zeta(2N+2-\lambda)\, \Gamma(2N+2-\lambda) + (2\pi)^{-2} \zeta(2N+4-\lambda) \\ \times \Gamma(2N+4-\lambda)].\end{aligned} \tag{7·21}$$

Substituting (7·13), (7·16), (7·18) and (7·19) in (7·2), we finally obtain the asymptotic expansion of $\Xi(\lambda, m)$ for $0 < \lambda < 2$ $(\lambda \neq 1)$:

$$\begin{aligned}\Xi(\lambda, m) = & \tfrac{1}{4}\sec^2(\tfrac{1}{2}\lambda\pi)\, m^{-\lambda-1} + \zeta(\lambda)\, (\pi m)^{-2} \ln m + \zeta'(\lambda)\, (\pi m)^{-2} \\ & - 8(2\pi)^{\lambda-1} \{\sin(\tfrac{1}{2}\lambda\pi)\, \zeta(3-\lambda)\, \Gamma(3-\lambda)\, (2\pi m)^{-4} \ln m + [\tfrac{1}{2}\pi \cos\tfrac{1}{2}\lambda\pi \\ & + \sin\tfrac{1}{2}\lambda\pi\, (\ln(2\pi) - \zeta'(3-\lambda)/\zeta(3-\lambda) - \Gamma'(3-\lambda)/\Gamma(3-\lambda))]\, \zeta(3-\lambda)\, \Gamma(3-\lambda)\, (2\pi m)^{-4} \\ & - \sin(\tfrac{1}{2}\lambda\pi)\, \zeta(5-\lambda)\, \Gamma(5-\lambda)\, (2\pi m)^{-6} \ln m - [\tfrac{1}{2}\pi \cos\tfrac{1}{2}\lambda\pi + \sin\tfrac{1}{2}\lambda\pi\, (\ln(2\pi) \\ & - \zeta'(5-\lambda)/\zeta(5-\lambda) - \Gamma'(5-\lambda)/\Gamma(5-\lambda))]\, \zeta(5-\lambda)\, \Gamma(5-\lambda)\, (2\pi m)^{-6} + \ldots\}.\end{aligned} \tag{7·22}$$

According to (7·1), this determines the asymptotic expansion of $\Xi(\lambda, m)$ for all positive non-integral values of λ.

In § 4 (*b*), we have to evaluate $\Xi(\lambda, m)$ and $\Xi'_\lambda(\lambda, m)$ for $\lambda = 1$. We cannot make $\lambda = 1$ in (7·22), since $\zeta(\lambda)$ has a simple pole at this point. However, since $\Xi(\lambda, m)$ is regular for $\lambda = 1$, we can develop the second member of (7·22) in a power series about this point. For

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this purpose, we need the Laurent series expansion of $\zeta(\lambda)$ about $\lambda = 1$, which is given by (Hardy 1912; Kluyver 1927)

$$\zeta(\lambda) = (\lambda-1)^{-1}+C+\sum_{n=1}^{\infty} C_n(\lambda-1)^n, \tag{7·23}$$

where C is Euler's constant, and

$$C_n = \frac{(-1)^n}{n!}\lim_{N\to\infty}\left[\sum_{v=1}^{N}\frac{(\ln v)^n}{v}-\frac{(\ln N)^{n+1}}{n+1}\right]. \tag{7·24}$$

These 'generalized Euler constants' may be evaluated numerically with the help of the Euler–Maclaurin sum formula, by the same method which is employed to evaluate Euler's constant (Valiron 1948, p. 219). The results are

$$C_1 = 0{\cdot}07282;\ C_2 = -0{\cdot}00486. \tag{7·25}$$

Making $\lambda = 1+\epsilon, |\epsilon| \ll 1$, in (7·22), and noticing that

$$m^{-\lambda-1} = m^{-2}[1-\epsilon\ln m+\tfrac{1}{2}\epsilon^2(\ln m)^2+O(\epsilon^3)], \tag{7·26}$$

we get, with the help of (7·23),

$$\begin{aligned}\Xi(1+\epsilon,m) = {} & \tfrac{1}{2}(\pi m)^{-2}(\ln m)^2+C(\pi m)^{-2}\ln m+(\tfrac{1}{12}\pi^2+C_1)(\pi m)^{-2}\\ & -[\tfrac{1}{6}(\pi m)^{-2}(\ln m)^3+(\tfrac{1}{12}\pi^2-C_1)(\pi m)^{-2}\ln m-2C_2(\pi m)^{-2}]\,\epsilon+O(\epsilon^2)\\ & -8(2\pi)^{\epsilon}[\cos(\tfrac{1}{2}\pi\epsilon)\,\zeta(2-\epsilon)\,\Gamma(2-\epsilon)\,(2\pi m)^{-4}\ln m+O(m^{-4})],\end{aligned} \tag{7·27}$$

from which it follows that

$$\begin{aligned}\Xi(1,m) = {} & \tfrac{1}{2}(\pi m)^{-2}(\ln m)^2+C(\pi m)^{-2}\ln m+(\tfrac{1}{12}\pi^2+C_1)(\pi m)^{-2}\\ & -8\zeta(2)\,\Gamma(2)\,(2\pi m)^{-4}\ln m+\ldots,\end{aligned} \tag{7·28}$$

$$\begin{aligned}-\Xi'_{\lambda}(1,m) = {} & \tfrac{1}{6}(\pi m)^{-2}(\ln m)^3+(\tfrac{1}{12}\pi^2-C_1)(\pi m)^{-2}\ln m-2C_2(\pi m)^{-2}\\ & +8\zeta(2)\,\Gamma(2)\,[\ln(2\pi)-\zeta'(2)/\zeta(2)-\Gamma'(2)/\Gamma(2)](2\pi m)^{-4}\ln m+O(m^{-4}).\end{aligned} \tag{7·29}$$

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